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A Uniform Asymptotic Formula for Orthogonal Polynomials Associated with $\exp(-x^4)$*

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Let $p_n(x)$ denote the orthogonal polynomials associated with the Freud weight $\exp(-x^4)$, $x \in R$. Let $x = (4n/3)^{1/4} w$. An asymptotic approximation is constructed for $p_n(x)$, which holds uniformly for $-1+\varepsilon \leqslant w \leqslant M$, where $0 < \varepsilon < 1$ and $1 < M < \infty$. This approximation involves the Airy function and its derivative, and it includes the two asymptotic formulas previously obtained by P. Nevai. Also presented is a four-term asymptotic expansion for the zeros of $p_n(x)$.

Key Words: orthogonal polynomials; Freud weight; turning; point; uniform asymptotic approximation; zeros.

1. INTRODUCTION

In [6], Nevai has studied the asymptotic behavior of the orthogonal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

associated with the weight function $\exp(-x^4)$. Here, the interval of orthogonality is the real line $\mathbb{R}$. These polynomials satisfy the recurrence relation

$$x p_n(x) = a_{n+1}\, p_{n+1}(x) + a_n p_{n-1}(x), \qquad n = 0, 1, ..., \tag{1.1}$$

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with $p_0(x) = \gamma_0 > 0$ and $p_1(x) = \gamma_0 x / a_1$. The coefficients a_n are determined successively from the equation

$$n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2), \qquad n = 1, 2, ..., \tag{1.2}$$

where $a_0^2 = 0$ and $a_1^2 = \Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})$. A two-term asymptotic expansion for a_n has been given by Lew and Quarles [2]. They showed that

$$a_n^2 = \left(\frac{n}{12}\right)^{1/2} \left[1 + \frac{1}{24n^2} + O\left(\frac{1}{n^4}\right)\right], \qquad n \to \infty. \tag{1.3}$$

If we let

$$\phi_n(x) = a_{n+1}^2 + a_n^2 + x^2, \tag{1.4}$$

then Shohat [9] and Nevai [5] independently showed that the function

$$z(x) = p_n(x)[\phi_n(x)]^{-1/2} \exp\left(-\frac{x^4}{2}\right) \tag{1.5}$$

satisfies the differential equation

$$z'' + f(n, x) z = 0, \tag{1.6}$$

where

$$\begin{aligned} f(n, x) = {} & 4a_n^2[4\phi_n(x)\,\phi_{n-1}(x) + 1 - 4a_n^2 x^2 - 4x^4 - 2x^2\phi_n(x)^{-1}] \\ & -4x^6 - 4x^4\phi_n(x)^{-1} - 3x^2\phi_n(x)^{-2} + 6x^2 + \phi_n(x)^{-1}. \end{aligned} \tag{1.7}$$

Based on the differential equation (1.6), Nevai [5] first obtained the asymptotic formula

$$\begin{aligned} & \exp\left(-\frac{x^4}{2}\right) p_n(x) \\ & \quad = A n^{-1/8} \sin\left\{\left(\frac{64}{27}\right)^{1/4} n^{3/4} x + (12)^{-1/4}\, n^{1/4} x^3 - (n-1)\frac{\pi}{2}\right\} + o(n^{-1/8}), \end{aligned} \tag{1.8}$$

holding uniformly for x in any finite interval, where A is a positive constant. To improve this result, Nevai [6] next considered the problem of deriving a Plancherel–Rotach type asymptotic formula for these polynomials $p_n(x)$.

More precisely, he showed that with $x=(4n/3)^{1/4}\cos\theta$ the asymptotic formula

$$p_n(x)\exp\left(-\frac{x^4}{2}\right)=12^{1/8}\pi^{-1/2}n^{-1/8}(\sin\theta)^{-1/2}\times\cos\left[\frac{n}{12}(12\theta-4\sin 2\theta-\sin 4\theta)+\frac{\theta}{2}-\frac{\pi}{4}\right]+O(n^{-9/8}) \tag{1.9}$$

holds uniformly for $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$

Note that the validity of formula (1.9) requires θ to be bounded away from 0 and π. In [3, p. 216], Lubuisky has obtained asymptotic formulas which hold uniformly for θ in intervals of the form $n^{-\delta}<\theta<\pi-n^{-\delta}$, $\delta>0$. Even this extension excludes the possibility of allowing $\theta=0$ or π. The aim of this paper is to present an asymptotic formula which is uniformly valid in an interval containing the critical value $\theta=0$. (It will become evident from our discussion that a corresponding result can be obtained for the case $\theta=\pi$ by using symmetry.) Our anticipated result corresponds to the classical Hilb formula for the Legendre polynomials [10, p. 197] or Erdélyi's asymptotic forms for the Laguerre polynomials [1].

As an application of our result, we shall also derive an asymptotic expansion for the large zeros of $p_n(x)$. Let us denote the zeros by $x_{n,n}<\cdots<x_{n,2}<x_{n,1}$. Máte, Nevai, and Totik [4] have already shown that

$$x_{n,1}=\left(\frac{4n}{3}\right)^{1/4}-\frac{i_1}{3\cdot 2^{1/3}}\left(\frac{4n}{3}\right)^{-5/12}+o(n^{-5/12}), \tag{1.10}$$

where i_1 is the smallest positive zero of Airy's function $A(x)$ defined in [10, p. 18, 377]. The result that we shall prove is

$$x_{n,k}=\left(\frac{4n}{3}\right)^{1/4}+\frac{\tilde{a}_k}{18^{1/3}}\left(\frac{4n}{3}\right)^{-5/12}+\frac{1}{6}\left(\frac{4n}{3}\right)^{-9/12}-\frac{19\tilde{a}_k^2}{90\cdot 2^{2/3}\cdot 3^{1/3}}\left(\frac{4n}{3}\right)^{-13/12}+O(n^{-17/12}), \tag{1.11}$$

for each fixed k, where $\tilde{a}_k$ is the kth negative zero of the usual Airy integral $Ai(x)$ defined in [8, pp. 53, 403]. In view of the relationship $\tilde{a}_1=-i_1/3^{1/3}$, (1.11) agrees with (1.10). Here we have used $\tilde{a}_k$, instead of a_k, to denote the zeros of $Ai(x)$, in order not to confuse with the coefficients in the recurrence relation (1.1). Note the (1.11) is not uniformly valid with respect to k.

2. TRANSFORMATION TO CANONICAL FORM

Recall that formula (1.9) holds for $0<\theta<\pi$ but fails to hold when $\theta=0$. Thus, $x=(4n/3)^{1/4}$ is a critical value for the validity of (1.9). Also note that (1.10)–(1.11) suggests that for sufficiently large values of n, all zeros of $p_n(x)$ lie in the interval $|x|<(4n/3)^{1/4}$. Based on these observations, we make the change of variable

$$x=\lambda w \qquad \text{with} \quad \lambda=\left(\frac{4n}{3}\right)^{1/4}. \tag{2.1}$$

Equation (1.6) then becomes

$$\frac{d^2z}{dw^2}+q(\lambda, w)\, z=0, \tag{2.2}$$

where

$$q(\lambda, w)=\lambda^2 f(n, \lambda w). \tag{2.3}$$

It is easily seen that the only singularities of $f(n, x)$ are the zeros of $\phi_n(x)$, which occur at $x=\pm i\sqrt{a_{n+1}^2+a_n^2}\approx \pm i(n/3)^{1/4}$. Hence, $q(\lambda, w)$ is analytic for w on the real axis.

LEMMA. *As $\lambda\to\infty$, $q(\lambda, w)$ has the asymptotic expansion*

$$q(\lambda, w)=\lambda^8\left[q_0(w)+\frac{q_4(w)}{\lambda^4}+\frac{q_8(w)}{\lambda^8}+\cdots\right] \tag{2.4}$$

which holds uniformly with respect to bounded w, where $q_0(w)=(1+3w^2-4w^6)$, $q_4(w)=(1+2w^2)$ and

$$q_8(w)=\frac{20w^4-64w^2+17}{9(1+2w^2)^2}.$$

Proof. Substitute (1.4) in (1.7) to give

$$f(n, \lambda w)=4a_n^2\left\{4(a_{n+1}^2+a_n^2)(a_n^2+a_{n-1}^2)+4\lambda^2w^2(a_{n+1}^2+a_n^2+a_{n-1}^2)+1\right.$$
$$\left.-\frac{2\lambda^2w^2}{a_{n+1}^2+a_n^2+\lambda^2w^2}\right\}-4\lambda^6w^6-\frac{4\lambda^4w^4}{a_{n+1}^2+a_n^2+\lambda^2w^2}$$
$$-\frac{3\lambda^2w^2}{(a_{n+1}^2+a_n^2+\lambda^2w^2)^2}+6\lambda^2w^2+\frac{1}{a_{n+1}^2+a^2-n+\lambda^2w^2}.$$

From (1.3), we have

$$(a_{n+1}^2+a_n^2)(a_n^2+a_{n-1}^2)=\frac{\lambda^4}{4}\left[1-\frac{5}{27}\frac{1}{\lambda^8}+O\left(\frac{1}{\lambda^{16}}\right)\right],$$

$$a_{n+1}^2+a_n^2+a_{n-1}^2=\frac{3}{4}\lambda^2\left[1-\frac{2}{27}\frac{1}{\lambda^8}+O\left(\frac{1}{\lambda^{16}}\right)\right],$$

and

$$a_{n+1}^2+a_n^2+\lambda^2 w^2=\frac{\lambda^2}{2}\left[(1+2w^2)+\frac{1}{3}\frac{1}{\lambda^4}-\frac{1}{27}\frac{1}{\lambda^8}+O\left(\frac{1}{\lambda^{16}}\right)\right].$$

A combination of the last four equations leads to

$$f(n,\lambda w)=(1+3w^2-4w^6)\,\lambda^6+(1+2w^2)\,\lambda^2 +\frac{20w^4-64w^2+17}{9(1+2w^2)^2}\frac{1}{\lambda^2}+O\left(\frac{1}{\lambda^6}\right),$$

where the O-term is uniform with respect to all bounded w. On account of (2.3), this establishes the lemma. ∎

Let us now rewrite (2.2) and (2.3) in the form

$$\frac{d^2z}{dw^2}=\Lambda^2\left[\bar{q}_0(w)+\frac{\bar{q}_1(w)}{\Lambda}+\frac{\bar{q}_2(w)}{\Lambda^2}+\cdots\right]z, \tag{2.5}$$

where $\Lambda=\lambda^4=4n/3$, $\bar{q}_0(w)=-q_0(w)=(4w^6-3w^2-1)=(2w^2+1)^2\,(w^2-1)$, $\bar{q}_1(w)=-q_4(w)=-(1+2w^2)$ and

$$\bar{q}_2(w)=-q_8(w)=-\frac{20w^4-64w^2+17}{9(1+2w^2)^2}.$$

Since $\bar{q}_0(w)$ vanishes at $w=\pm 1$ and the other coefficient functions $\bar{q}_s(w)$ are analytic there, we have exactly the extended form of the turning point problem discussed in [8, p. 426]. From the recurrence relation (1.1), one can readily verify by induction that $p_n(x)$ satisfies the reflection formula

$$p_n(x)=(-1)^n\,p_n(-x).$$

Thus we need consider only the turning point $w=+1$. As suggested in [8, p. 426], we make the Liouville–Green transformations

$$\zeta=\left\{\frac{3}{2}\int_1^w \bar{q}_0^{1/2}(t)\,dt\right\}^{2/3}, \qquad Z=\left(\frac{d\zeta}{dw}\right)^{1/2} z=\hat{q}^{1/4}(w)\,z, \tag{2.6}$$

where

$$\hat{q}(w) = \frac{\bar{q}_0(w)}{\zeta} = \frac{(2w^2+1)^2\,(w^2-1)}{\zeta}. \tag{2.7}$$

Evaluation of the integral in (2.6) gives

$$\zeta(w) = \begin{cases} -[\frac{9}{8}\cos^{-1} w - \frac{3}{8}\, w(2w^2+1)\sqrt{1-w^2}\,]^{2/3}, & -1 < w < 1 \\ [\frac{3}{8}\, w(2w^2+1)\sqrt{w^2-1} - \frac{9}{8}\cosh^{-1} w]^{2/3}, & w \geqslant 1. \end{cases} \tag{2.8}$$

It can be shown that the quantities inside the square brackets in (2.8) are positive. Equation (2.5) then becomes

$$\frac{d^2 Z}{d\zeta^2} = \{ \Lambda^2 \zeta + \Lambda \eta(\zeta) + \psi(\Lambda, \zeta) \}\, Z, \tag{2.9}$$

with

$$\eta(\zeta) = \frac{\bar{q}_1(w)}{\hat{q}(w)} = \frac{-\zeta}{(2w^2+1)(w^2-1)} \tag{2.10}$$

and

$$\psi(\Lambda, \zeta) \sim \sum_{s=0}^{\infty} \frac{\psi_s(\zeta)}{\Lambda^s}. \tag{2.11}$$

By applying l'Hôpital's rule to (2.10), it can be shown that $\eta(0) = -3^{-1/3} 2^{-2/3}$; cf. (2.6). Simple calculation also gives

$$\psi_0(\zeta) = \frac{\bar{q}_2(w)}{\hat{q}(w)} - \frac{1}{\hat{q}^{3/4}(w)} \frac{d^2}{dw^2} \left\{ \frac{1}{\hat{q}^{1/4}(w)} \right\}$$

and $\psi_s(\zeta) = \bar{q}_{s+2}(w)/\hat{q}(w)$. Since $\zeta(w)$, $\bar{q}_s(w)$ and $1/\hat{q}(w)$ are all analytic in a region containing the real axis in the w-plane, the functions $\eta(\zeta)$ and $\psi_s(\zeta)$ are analytic on the real line. In [8, p. 427], Olver has shown that (2.9) is formally satisfied by the series

$$Z_1(\Lambda, \zeta) \sim Ai\left(\Lambda^{2/3}\zeta + \frac{H(\zeta)}{\Lambda^{1/3}} \right) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\Lambda^s} + \frac{1}{\Lambda^{4/3}}\, Ai'\left(\Lambda^{2/3}\zeta + \frac{H(\zeta)}{\Lambda^{1/3}} \right) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\Lambda^s}, \tag{2.12}$$

where

$$H(\zeta) = \frac{1}{2\zeta^{1/2}} \int_0^{\zeta} \frac{\eta(v)}{v^{1/2}}\, dv \tag{2.13}$$

and the coefficients A_s, and B_s satisfy the recurrence relations

$$\begin{aligned} B_s + 2\zeta B_s' = \psi_0 A_s + \psi_1 A_{s-1} + \cdots \\ &+ \psi_s A_0 - (2HH' + \zeta H'^2)\, A_s - A_s'' - HH'^2 A_{s-1} \\ &- (2H' + \zeta H'')\, B_{s-1} - 2(\zeta H' + H)\, B_{s-1}' \\ &- (HH'' + H'^2)\, B_{s-2} - 2HH' B_{s-2}' \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} 2A_{s+1}' = -H'' A_s - 2H' A_s' + \psi_0 B_{s-1} + \psi_1 B_{s-2} + \cdots + \psi_{s-1} B_0 \\ - (2HH' + \zeta H'^2)\, B_{s-1} - B_{s-1}'' - HH'^2 B_{s-2}, \end{aligned} \tag{2.15}$$

with $A_0(\zeta) = 1$ and

$$B_0(\zeta) = \frac{1}{2\zeta^{1/2}} \int_0^\zeta \{\psi_0(v) - 2H(v)\, H'(v) - vH'^2(v)\} \frac{dv}{v^{1/2}}.$$

Note that by inserting (2.10) in (2.13) and using (2.6), it can be proved that

$$H(\zeta) = \begin{cases} \dfrac{-1}{2\zeta^{1/2}} \cosh^{-1} w, & w \geqslant 1 \\ \dfrac{-1}{2(-\zeta)^{1/2}} \cos^{-1} w, & |w| < 1 \end{cases} \tag{2.16}$$

Also note that by applying l'Hôpital's rule directly to (2.13), it can be verified that

$$H(0) = \frac{-1}{3^{1/3} 2^{2/3}} \qquad \text{and} \qquad H'(0) = \frac{1}{15}. \tag{2.17}$$

Since $2\zeta H'(\zeta) + H(\zeta) = \eta(\zeta)$ by (2.13), it follows from (2.15)

$$\begin{aligned} A_1(\zeta) &= -\frac{1}{2} \int_0^\zeta H''(\zeta)\, d\zeta = -\frac{1}{2} [H'(\zeta) - H'(0)] \\ &= -\frac{1}{2} \left\{ \frac{\eta(\zeta) - H(\zeta)}{2\zeta} - \frac{1}{15} \right\}. \end{aligned}$$

Olver [8, p. 428] has also briefly indicated how to establish the asymptotic nature of the formal expansion (2.12). Along the lines he has suggested, we have obtained an analogue of Theorem 7.1 in [8, p.410]. Here we shall be content with just the special case of a one-term approximation. Before stating the result, we first recall the modulus function $M(x)$ and the weight function $E(x)$ associated with the Airy functions $Ai(x)$ and $Bi(x)$; cf. [8, p. 395]. Let $x=c$ denote the negative root of the equation

$$Ai(x)=Bi(x)$$

of smallest absolute value. Numerical calculation has shown that $c=-0.36605$, correct to five decimal places. Define $E(x)=1$ for $-\infty<x\leqslant c$,

$$E(x)=\{Bi(x)/Ai(x)\}^{1/2}, \qquad c\leqslant x<\infty,$$

and

$$M(x)=\{E^2(x)\,Ai^2(x)+E^{-2}(x)\,Bi^2(x)\}^{1/2},$$

where $E^{-1}(x)=1/E(x)$. Next we introduce the error control function

$$\Phi_0(\Lambda,\zeta)=\int_0^\zeta\left|\xi+\frac{H(\xi)}{\Lambda}\right|^{-1/2}d\xi. \tag{2.18}$$

It is readily seen that this integral is convergent.

THEOREM 1. *Equation* (2.9) *has a pair of infinitely differentiable solution* $Z_1(\Lambda,\zeta)$ *and* $Z_2(\Lambda,\zeta)$, *given by*

$$Z_1(\Lambda,\zeta)=\left\{Ai\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)+\varepsilon_1(\Lambda,\zeta)\right\}, \tag{2.19}$$

$$Z_2(\Lambda,\zeta)=\left\{Bi\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)+\varepsilon_2(\Lambda,\zeta)\right\}, \tag{2.20}$$

For sufficiently large value of Λ, *the error terms satisfy*

$$|\varepsilon_1(\Lambda,\zeta)|/M\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right), \qquad |\partial\varepsilon_1(\Lambda,\zeta)/\partial\zeta|/\Lambda^{2/3}N\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)$$
$$\leqslant\frac{K}{\Lambda}E^{-1}\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)\exp\left\{\frac{K_0}{\Lambda}\mathcal{V}_{\zeta,\zeta(M)}(|\zeta|^{1/2}B_0(\zeta))\right\}\mathcal{V}_{\zeta,\zeta(M)}(\Phi_0(\Lambda,\zeta)) \tag{2.21}$$

and

$$|\varepsilon_2(\Lambda,\zeta)|/M\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right),\ |\partial\varepsilon_2(\Lambda,\zeta)/\partial\zeta|/\Lambda^{2/3}N\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)$$
$$\leqslant\frac{K}{\Lambda}E\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)\exp\left\{\frac{K_0}{\Lambda}\mathcal{V}_{\zeta(-1+\varepsilon),\zeta}(|\zeta|^{1/2}\,B_0(\zeta))\right\}$$
$$\times\mathcal{V}_{\zeta(-1+\varepsilon),\zeta}(\Phi_0(\Lambda,\zeta)) \tag{2.22}$$

where K and K_0 are positive constants, $\mathcal{V}_{a,b}(f)$ denotes the total variation of a function $f(x)$ on an interval (a, b), and $\zeta = \zeta(w)$ is the function given in (2.8).

The total variation of $f(x)$ on $[a, b]$ is defined by

$$\mathcal{V}_{a,b}(f) = \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all possible subdivisions of the interval $[a, b]$. If $f(x)$ is continuously differentiable in $[a, b]$, then we also have

$$\mathcal{V}_{a,b}(f) = \int_a^b |f'(x)|\, dx;$$

see [8, pp. 27–28].

In view of asymptotic forms

$$E(x) \sim 2^{1/2} \exp(\tfrac{2}{3} x^{3/2}), \qquad M(x) \sim \pi^{-1/2} x^{-1/4} \qquad (x \to +\infty)$$

and

$$M(x) \sim \pi^{-1/2}(-x)^{-1/4} \qquad (x \to -\infty),$$

we have

$$\varepsilon_1(\Lambda, \zeta) = O(\Lambda^{-1}), \tag{2.23}$$

as $\Lambda \to \infty$, uniformly with respect to ζ in any bounded interval containing $\zeta = 0$ and contained in $(\zeta(-1), \infty)$. When ζ is bounded away from 0, we also have

$$\varepsilon_1(\Lambda, \zeta) = O(\Lambda^{-7/6}). \tag{2.24}$$

3. UNIFORM ASYMPTOTIC FORMULA FOR $p_n(x) \exp(-x^4/2)$

We first recall the asymptotic formulas

$$Ai(x) \sim \frac{1}{2\pi^{1/2} x^{1/4}} \exp\left(-\frac{2}{3} x^{3/2}\right)$$

and

$$Bi(x) \sim \frac{1}{\pi^{1/2} x^{1/4}} \exp\left(\frac{2}{3} s^{3/2}\right)$$

as $x \to \infty$. Since the function $z(x) = p_n(x)[\phi_n(x)]^{-1/2} \exp(-x^4/2)$ in (1.5) is exponentially small as $x \to \infty$, by (2.6) there exists a constant $C(n)$ such that

$$p_n(x)[\phi_n(x)]^{-1/2} \exp\left(-\frac{x^4}{2}\right) = C(n)\, \hat{q}^{-1/4}(w)\, Z_1(\Lambda, \zeta), \tag{3.1}$$

where $Z_1(\Lambda, \zeta)$ is the asymptotic solution given in (2.12); cf. also (2.19). Next we need to find a formula for $C(n)$ as $n \to \infty$. When $x = 0$, we have $w = 0$ by (2.1) and $\zeta = -(9\pi/16)^{2/3}$ by (2.8). Since $\Lambda = 4n/3$, it follows from (3.1), (2.19), and (1.4) that

$$\begin{aligned} &p_n(0)[a_{n+1}^2 + a_n^2]^{-1/2} \\ &\quad = C(n)\, \hat{q}^{-1/4}(0) \left\{ Ai\left[-\left(\frac{3\pi}{4}\right)^{2/3} n^{2/3} - \left(\frac{\pi}{4\sqrt{3}}\right)^{2/3} \frac{1}{n^{1/3}} \right] + O(n^{-7/6}) \right\}. \end{aligned} \tag{3.2}$$

Here we have also made use of (2.16) and (2.24).

Since $p_n(x)$ satisfies the reflection formula mentioned in Section 2, it contains only odd powers of x when n is odd, and only even powers of x when n is even. Furthermore, $p_{2k+1}(0) = 0$ and

$$p_{2k}(0) = (-1)^k \gamma_0 \frac{a_1 a_3 \cdots a_{2k-1}}{a_2 a_4 \cdots a_{2k}}. \tag{3.3}$$

Since the infinite products

$$\prod_{k=1}^{\infty} \frac{12 a_{2k-1}^4}{2k-1} \qquad \text{and} \qquad \prod_{k=1}^{\infty} \frac{2k}{12 a_{2k}^4}$$

are convergent, from (3.3) it can be proved that when n is even, there exists a constant A such that

$$p_n(0) = A \cos\left(\frac{n\pi}{2}\right) n^{-1/8}[1 + O(n^{-1})]; \tag{3.4}$$

see [6]. Recall the asymptotic formula [8, p. 392]

$$Ai(-x) = \frac{1}{\pi^{1/2} x^{1/4}} \left\{ \cos\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4}\right) [1 + O(x^{-3})] \right.$$

$$\left. + \sin\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4}\right) \left[\frac{5}{48} x^{-3/2} + O(x^{-9/2})\right] \right\} \tag{3.5}$$

as $x \to +\infty$. Upon simplification, we obtain

$$Ai\left[-\left(\frac{3\pi}{4}\right)^{2/3} n^{2/3} - \left(\frac{\pi}{4\sqrt{3}}\right)^{2/3} \frac{1}{n^{1/3}}\right]$$

$$= \left(\frac{3}{4}\right)^{-1/6} \pi^{-2/3} (-1)^{n/2} n^{-1/6} [1 + O(n^{-1})]. \tag{3.6}$$

Thus, when n is an even integer, a combination of (3.2), (3.4), and (3.6) gives

$$C(n) = 3^{1/12} 2^{1/3} \pi^{1/2} A n^{-5/24} [1 + O(n^{-1})] \tag{3.7}$$

on account of (1.3) and (2.7). Note that $\hat{q}(0) = (16/9\pi)^{2/3}$ and $\cos(n\pi/2) = (-1)^{n/2}$. Inserting (3.7) and (2.19) into (3.1) yields

$$p_n(x) [\phi_n(x)]^{-1/2} \exp\left(-\frac{x^4}{2}\right) = 3^{1/12} 2^{1/3} \pi^{1/2} A n^{-5/24} [1 + O(n^{-1})]$$

$$\times \hat{q}^{-1/4}(w) \left\{ Ai\left(\Lambda^{2/3} \zeta + \frac{H(\zeta)}{\Lambda^{1/3}}\right) + \varepsilon_1(\Lambda, \zeta) \right\}. \tag{3.8}$$

Since $\phi_n^{1/2}(x) = (n/3)^{1/4} (1 + 2w^2)^{1/2} [1 + O(n^{-1})]$ by (1.3) and (2.1), (3.8) becomes

$$p_n(x) \exp\left(-\frac{x^4}{2}\right) = 3^{-1/6} 2^{1/3} \pi^{1/2} A (1 + 2w^2)^{1/2} n^{1/24}$$

$$\times \hat{q}^{-1/4}(w) \left\{ Ai\left(\Lambda^{2/3} \zeta + \frac{H(\zeta)}{\Lambda^{1/3}}\right) + O(n^{-1}) \right\} \tag{3.9}$$

by virtue of (2.23), uniformly for $\zeta \in (\zeta(-1+\varepsilon), \zeta(M))$ or, equivalently, $-1+\varepsilon < w < M$. If ζ is bounded away from 0, i.e., w bounded away from 1, then we have even the better estimate

$$p_n(x) \exp\left(-\frac{x^4}{2}\right) = 3^{-1/6} 2^{1/3} \pi^{1/2} A (1+2w^2)^{1/2}\, n^{1/24} \times \hat{q}^{-1/4}(w) \left\{ Ai\left(\Lambda^{2/3}\zeta + \frac{H(\zeta)}{\Lambda^{1/3}}\right) + O(n^{-7/6}) \right\} \tag{3.10}$$

on account of (2.24).

When n is an odd integer, from (3.1) we must have $Z_1(\Lambda, \zeta(0)) = 0$ since $p_n(0) = 0$. Thus, the value of $C(n)$ in this case can be determined from (3.1) by a limiting process. Put

$$C(n) = \lim_{x \to 0} \frac{p_n(x)[\phi_n(x)]^{-1/2} \exp(-x^4/2)}{\hat{q}^{-1/4}(w)\, Z_1(\Lambda, \zeta(w))}, \tag{3.11}$$

and observe that

$$\lim_{x \to 0} [\phi_n(x)]^{-1/2} \exp(-x^4/2)\, \hat{q}^{1/4}(w) = (a_{n+1}^2 + a_n^2)^{-1/2} \left(\frac{16}{9\pi}\right)^{1/6} = 2^{2/3} \pi^{-1/6} 3^{-1/12} n^{-1/4} [1 + O(n^{-1})]. \tag{3.12}$$

By l'Hôpital's rule,

$$\lim_{x \to 0} \frac{p_n(x)}{Z_1(\Lambda, \zeta(w))} = \lim_{x \to 0} \frac{p_n'(x)}{(dZ_1/d\zeta)(d\zeta/dw)(dw/dx)}. \tag{3.13}$$

Since

$$\left.\frac{dw}{dx}\right|_{x=0} = \left(\frac{4n}{3}\right)^{-1/4}, \tag{3.14}$$

$$\left.\frac{d\zeta}{dw}\right|_{w=0} = \left(\frac{16}{9\pi}\right)^{1/3} \tag{3.15}$$

and

$$\frac{dZ_1}{d\zeta} = Ai'\left(\Lambda^{2/3}\zeta + \frac{H(\zeta)}{\Lambda^{1/3}}\right)\left(\Lambda^{2/3} + \frac{H'(\zeta)}{\Lambda^{1/3}}\right) + \frac{d\varepsilon_1}{d\zeta}, \tag{3.16}$$

from (3.13) we obtain

$$\lim_{x\to 0}\frac{p_n(x)}{Z_1(\Lambda,\zeta(w))}=(-1)^{-(n-1)/2}\,2^{-11/6}3^{11/12}\pi^{2/3}p_n'(0)\,n^{-7/12}[1+O(n^{-1})], \tag{3.17}$$

where we have made use of the facts that $C(0)=-(9\pi/16)^{2/3}$, $\Lambda=4n/3$ and

$$Ai'(-x)=\frac{x^{1/4}}{\pi^{1/2}}\sin\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)+O(x^{-5/4}),\qquad x\to\infty; \tag{3.18}$$

see [8, p. 392]. Inserting (3.12) and (3.17) into (3.11) gives

$$C(n)=(-1)^{-(n-1)/2}\,2^{-7/6}3^{5/6}\pi^{1/2}n^{-5/6}p_n'(0)[1+O(n^{-1})].$$

Using (1.1), (1.2) and (1.4), it can be verified by induction that

$$p_n'(0)=4a_n\phi_n(0)\,p_{n-1}(0);$$

see also [5, formula (12)]. Coupling the last two equations yields

$$C(n)=3^{1/12}2^{1/3}\pi^{1/2}An^{-5/24}[1+O(n^{-1})]$$

on account of (3.4), which is exactly the same as (3.7). Therefore,

$$C(n)\sim 3^{1/12}2^{1/3}\pi^{1/2}An^{-5/24}$$

for all $n\geqslant 1$, and the asymptotic formulas (3.9) and (3.10) hold for all n, whether n is even or odd.

The value of the constant A in (3.4) has already been given explicitly by Nevai [6, p. 1183]:

$$A^2=12^{1/4}/\pi. \tag{3.19}$$

However, by using the uniform asymptotic formula (3.9), a shorter proof can be provided.

THEOREM 2. *Let $0<\varepsilon<1$ and $0<M<\infty$ be fixed, and let $x=(4n/3)^{1/4}w$ and $\Lambda=4n/3$. Then the asymptotic formula*

$$p_n(x)\exp\left(-\frac{x^4}{2}\right)=\sqrt{2}\,\Lambda^{1/24}\left(\frac{\zeta}{w^2-1}\right)^{1/4}\left\{Ai\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)+O(n^{-1})\right\} \tag{3.20}$$

holds uniformly for $-1+\varepsilon \leqslant w \leqslant M$, *where* ζ *and* $H(\zeta)$ *are given in* (2.8) *and* (2.16), *respectively. Moreover, when* $-1+\varepsilon \leqslant w \leqslant 1-\varepsilon$, *we also have the uniform asymptotic formula*

$$p_n(x) \exp\left(-\frac{x^4}{2}\right) = \sqrt{2}\, \Lambda^{1/24} \left(\frac{\zeta}{w^2-1}\right)^{1/4} \left\{ Ai\left(\Lambda^{2/3}\zeta + \frac{H(\zeta)}{\Lambda^{1/3}}\right) + O(n^{-7/6})\right\} \tag{3.21}$$

To see that Nevai's result (1.9) can be deduced from (3.21), we note from (3.5) that

$$p_n(x) \exp\left(-\frac{x^4}{2}\right) = \frac{12^{1/8}}{\pi^{1/2}}\, n^{-1/8} \frac{1}{(1-w^2)^{1/4}} \times \cos\left\{\frac{8}{9}(-\zeta)^{3/2}\, n - \sqrt{-\zeta}\, H(\zeta) - \frac{\pi}{4}\right\} + O(n^{-9/8}) \tag{3.22}$$

for $-1+\varepsilon \leqslant w \leqslant 1-\varepsilon$. Let $w = \cos\theta$. Then from (3.22) we obtain

$$p_n(x) \exp\left(-\frac{x^4}{2}\right) = 12^{1/8}\pi^{-1/2} n^{-1/8} (\sin\theta)^{-1/2} \times \cos\left\{n\theta - \frac{n}{3}\cos\theta(2\cos^2\theta+1)\sin\theta + \frac{\theta}{2} - \frac{\pi}{4}\right\} + O(n^{-9/8}) \tag{3.23}$$

uniformly for $\bar{\varepsilon} \leqslant \theta \leqslant \pi - \bar{\varepsilon}$, $\bar{\varepsilon} > 0$. In view of a trigonometric identity, it is readily seen that (3.23) agrees with (1.9).

4. PROOF OF (1.11)

Let the zeros of $p_n(x)$ be arranged in decreasing order:

$$-\infty < x_{n,n} < x_{n,n-1} < \cdots < x_{n,2} < x_{n,1} < \infty.$$

In view of the fact that

$$\lim_{w \to 1} \frac{\zeta}{w^2-1} = \left(\frac{3}{2}\right)^{2/3} \neq 0,$$

it follows from (3.20) (see also (3.8)) that $x_{n,k}$ can be determined by the roots of the equation

$$Ai\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right]+\varepsilon_1(\Lambda,\zeta)=0, \tag{4.1}$$

where $\varepsilon_1(\Lambda,\zeta)$ satisfies the estimates in (2.2). Let $\tilde{a}_k$ denote the kth negative zero of the Airy integral $Ai(x)$, and let $\zeta_{n,k}$ denote the kth root of equation (4.1). Since $\varepsilon_1(\Lambda,\zeta)=O(n^{-1})$, it is reasonable to expect that

$$\zeta_{n,k}\approx\zeta_n(\tilde{a}_k),$$

where $\zeta_n(\tilde{a}_k)$ satisfies

$$\left(\frac{4n}{3}\right)^{2/3}\zeta_n(\tilde{a}_k)+\frac{H(\zeta_n(\tilde{a}_k))}{(4n/3)^{1/3}}=\tilde{a}_k. \tag{4.2}$$

since $\zeta=\zeta(w)$ is finite on the interval $-1+\varepsilon\leqslant w\leqslant M$, $H(\zeta)$ is bounded on $\zeta(-1+\varepsilon)\leqslant\zeta\leqslant\zeta(M)$. From (4.2), it follows that $\zeta_n(\tilde{a}_k)$ is negative when n sufficiently large.

In the following analysis, we suppose that Λ is large enough so that

$$\frac{K}{\Lambda}\mathscr{V}_{\zeta,\zeta(M)}(\Phi_0(\Lambda,\zeta))\exp\left\{\frac{K_0}{\Lambda}\mathscr{V}_{\zeta,\zeta(M)}(|\zeta|^{1/2}B_0)\right\}<\frac{1}{2}. \tag{4.3}$$

Also, we put

$$p_1(\Lambda,\zeta)=\varepsilon_1(\Lambda,\zeta)\,E\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right]\Big/M\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right] \tag{4.4}$$

and

$$\sigma_1(\Lambda,\zeta)=\frac{K}{\Lambda}\mathscr{V}_{\zeta,\zeta(M)}(\Phi_0(\Lambda,\zeta))\exp\left\{\frac{K_0}{\Lambda}\mathscr{V}_{\zeta,\zeta(M)}(|\zeta|^{1/2}\,B_0)\right\}. \tag{4.5}$$

From (2.21), we have $|\rho_1(\Lambda,\zeta)|\leqslant\sigma_1(\Lambda,\zeta)<\frac{1}{2}$. We shall next rewrite equation (4.1) in terms of the phase function $\theta(x)$ defined by

$$E(x)\,Ai(x)=M(x)\sin\theta(x),\qquad E^{-1}(x)\,Bi(x)=M(x)\cos\theta(x). \tag{4.6}$$

see [8, p. 394]. Note that

$$\theta(x)=\tan^{-1}\{E^2(x)\,Ai(x)/Bi(x)\}.$$

and $\theta(x)=\frac{1}{4}\pi$ for $x \geqslant c$; cf. [8, p. 395]. Coupling (4.1) and (4.6), we obtain

$$\sin\theta\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right]=-\rho_1(\Lambda,\zeta). \tag{4.7}$$

Since the left-hand side equals $1/\sqrt{2}$ when $\Lambda^{2/3}\zeta+H(\zeta)\,\Lambda^{-1/3}\geqslant c$ while the right-hand side is less than 1/2, there can be no roots to equation (4.1) in this range. Hence, for $\zeta\geqslant 0$ and Λ sufficiently large, the function on the left-hand side of (4.1) has no zeros. Recall that $\zeta\geqslant 0$ corresponds to $w\geqslant 1$ by (2.8), and that $w\geqslant 1$ corresponds to $x\geqslant(4n/3)^{1/4}$ by (2.1). Therefore, in view of (3.20), the polynomial $p_n(x)$ has no zero in $x\geqslant(4n/3)^{1/4}$. By symmetry, $p_n(x)$ also has no zero in $x\leqslant-(4n/3)^{1/4}$. That is, all zeros of $p_n(x)$ lie in the interval $-(4n/3)^{1/4}<x<(4n/3)^{1/4}$; equivalently, all roots of equation (4.7) lie in the interval $-(9\pi/8)^{2/3}<\zeta<0$. In this range, the trigonometric equation (4.7) can be written in the form

$$\theta\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right]-k\pi-(-1)^{k-1}\arcsin\{\rho_1(\Lambda,\zeta)\}=0. \tag{4.8}$$

Let b_k denote the kth negative zero of $Bi(x)$, and let $\zeta_n(b_k)$ satisfy the equation

$$\left(\frac{4n}{3}\right)^{2/3}\zeta_n(b_k)+\frac{H(\zeta_n(b_k))}{(4n/3)^{1/3}}=b_k. \tag{4.9}$$

From the estimate

$$|\arcsin\rho_1(\Lambda,\zeta)|<\arcsin\frac{1}{2}=\frac{\pi}{6}$$

and the result [8, p. 404]

$$\theta(b_k)=(k-\tfrac{1}{2})\pi,$$

it is readily seen that the left-hand side of (4.8) is negative when $\zeta=\zeta_n(b_k)$ and positive when $\zeta=\zeta_n(b_{k+1})$. Hence, in the range

$$b_{k+1}<\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}<b_k \tag{4.10}$$

or, equivalently, $\zeta_n(b_{k+1})<\zeta<\zeta_n(b_k)$, Eq. (4.8) must have a root. [Note that $\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)$ is monotonically increasing in ζ for sufficiently large values of Λ.] Since the function on the left-hand side of (4.8) is decreasing in the interval $\zeta_n(b_{k+1})<\zeta<\zeta_n(b_k)$, as we shall see later in the discussion, Eq. (4.8) has only one root in this range. Let us now investigate the relationship between this zero and the kth zero of $Ai(x)$. By the mean-value theorem, we have

$$\theta\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right]=\theta(\tilde{a}_k)+\left[\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}-\tilde{a}_k\right]\theta'(\xi),$$

where $\xi\in(b_{k+1}, b_k)$. Recall from [8, p. 404] that $\theta(\tilde{a}_k)=k\pi$. Hence, by (4.8).

$$\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}-\tilde{a}_k=(-1)^{k-1}\arcsin\{\rho_1(\Lambda,\zeta)\}/\theta'(\xi). \tag{4.11}$$

Using the inequalities $|\rho_1(\Lambda,\zeta)|\leqslant\sigma_1(\Lambda,\zeta)<1/2$ and $\sin t>(3/\pi)t$ for $0<t<\pi/6$, we obtain

$$\left|\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}-\tilde{a}_k\right|\leqslant\frac{\pi}{3}\sigma_1(\Lambda,\zeta)/|\theta'(\xi)|.$$

Since $|\theta'(\xi)|$ is decreasing in ξ (see [8, p. 404]) and $\sigma_1(\Lambda,\zeta)$ is decreasing in ζ, it follows that

$$\left|\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}-\tilde{a}_k\right|\leqslant\alpha_k, \tag{4.12}$$

where

$$\alpha_k=\frac{\pi}{3}\sigma_1(\Lambda,\zeta_n(b_{k+1}))/|\theta'(b_k)|. \tag{4.13}$$

In view of the identity [8, p. 404]

$$\theta'(x)=1/\{\pi M^2(x)\}, \tag{4.14}$$

equation (4.13) gives

$$\alpha_k=\frac{\pi^2}{3}M^2(b_k)\,\sigma_1(\Lambda,\zeta_n(b_{k+1}))=M^2(b_k)\,O\left(\frac{1}{n}\right); \tag{4.15}$$

cf. (4.5).

We now turn to the proof of the monotonicity of the function on the left-hand side of (4.8). By virtue of (4.14) and the asymptotic formula [8, p. 395]

$$M(x)\sim\pi^{-1/2}x^{-1/4}\qquad(x\to+\infty),$$

$\theta'(x)$ is strictly negative for all $x>0$. Thus, to prove that the derivative of the function on the left-hand side of (4.8) is strictly less than zero, it suffices to show that

$$\left\{1-\rho_1^2(\Lambda,\zeta)\right\}^{-1/2}\left|\frac{d}{d\zeta}\rho_1(\Lambda,\zeta)\right|<\left[\Lambda^{2/3}+\frac{H'(\zeta)}{\Lambda^{1/3}}\right]\left|\theta'\left(\Lambda^{2/3}\zeta+\frac{H(\zeta)}{\Lambda^{1/3}}\right)\right|. \tag{4.16}$$

Also since $E(x)=1$ for $x\leqslant c$, on account of (4.4) we have

$$\frac{d}{d\zeta}\rho_1(\Lambda,\zeta)=\frac{1}{M[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]}\frac{d}{d\zeta}\varepsilon_1(\Lambda,\zeta)$$
$$-\frac{M'[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]}{M^2[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]}[\Lambda^{2/3}+\Lambda^{-1/3}H'(\zeta)]\,\varepsilon_1(\Lambda,\zeta).$$

From (2.21), it follows that

$$\left\{1-\rho_1^2(\Lambda,\zeta)\right\}^{-1/2}\left|\frac{d}{d\zeta}\rho_1(\Lambda,\zeta)\right|$$
$$\leqslant\frac{|\Lambda^{2/3}+\Lambda^{-1/3}H'(\zeta)|}{\{1-\sigma_1^2(\Lambda,\zeta)\}^{1/2}}\sigma_1(\Lambda,\zeta)$$
$$\times\frac{N[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]+M'[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]}{M[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]}. \tag{4.17}$$

Here we have also made use of the fact that $H'(\zeta)>0$ and hence $\Lambda^{2/3}<\Lambda^{2/3}+\Lambda^{-1/3}H'(\zeta)$. Coupling (4.17) and (4.14), it is evident that (4.16) holds if

$$\frac{\sigma_1(\Lambda,\zeta)}{\{1-\sigma_1^2(\Lambda,\zeta)\}^{1/2}}<\frac{1}{\left[\begin{array}{c}\pi M[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]\{N[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]\\ +M'[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)\}\end{array}\right]}. \tag{4.18}$$

The left-hand side of (4.18) is less than $1/\sqrt{3}=0.577...$, since $\sigma_1(\Lambda,\zeta)<\frac{1}{2}$. The right-hand side of (4.18) is decreasing function of $[\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)]$ by Lemma 5.1 in [8, p. 404]. When $\Lambda^{2/3}\zeta+\Lambda^{-1/3}H(\zeta)=c$, its value is

$$\frac{1}{\pi Ai(c)\{Ai'(c)+Bi'(c)+\sqrt{2Ai'^2(c)+2Bi'^2(c)}\}}=0.708....$$

Therefore, (4.18) and (4.16) are satisfied, and equation (4.8) has only one root in the interval (4.10) or, equivalently, in the interval $\zeta_n(b_{k+1}) < \zeta < \zeta_n(b_k)$. (Much of the above argument is patterned after that given in [8, pp. 406–407.)

Let $x_{n,k}$ denote the kth zero of $p_n(x)$, counted from right to left. Let $\zeta_{n,k}$ and $w_{n,k}$ denote the corresponding values determined by (2.1) and (2.8), respectively. Since a root of equation (4.1) is also a root of Eq. 4.8), $\zeta_{n,k}$ satisfies $\zeta_n(b_{k+1}) < \zeta_{n,k} < \zeta_n(b_k)$, or more accurately

$$\tilde{a}_k - \alpha_k < \Lambda^{2/3}\zeta_{n,k} + \frac{H(\zeta_{n,k})}{\Lambda^{1/3}} < \tilde{a}_k + \alpha_k$$

on account of (4.12). Therefore

$$\Lambda^{2/3}\zeta_{n,k} + \frac{H(\zeta_{n,k})}{\Lambda^{1/3}} = \tilde{a}_k + O\left(\frac{1}{n}\right); \qquad (4.19)$$

see (4.15). Since $H(\zeta_{n,k})$ is bounded for all n and k, we have the prelimary approximation $\zeta_{n,k} = \tilde{a}_k \Lambda^{-2/3} + O(n^{-1})$. By the mean-value theorem, $H(\zeta_{n,k}) = H(0) + O(n^{-2/3}) = -3^{-1/3}2^{-2/3} + O(n^{-2/3})$. Substituting this into (4.19) gives

$$\zeta_{n,k} = \tilde{a}_k \Lambda^{-2/3} + 3^{-1/3}2^{-2/3}\Lambda^{-1} + O(n^{-5/3}). \qquad (4.20)$$

Let $\Omega(\zeta)$ denote the inverse of the function $\zeta(w)$ so that $\zeta = \zeta(w)$ if and only if $w = \Omega(\zeta)$. Since $\zeta_{n,k} = \zeta(w_{n,k})$ and $x_{n,k} = (4n/3)^{1/4} w_{n,k}$, from (4.20) we derive

$$x_{n,k} = \left(\frac{4n}{3}\right)^{1/4} \Omega\left\{\left[\tilde{a}_k \left(\frac{4n}{3}\right)^{-2/3} + 3^{-1/3}2^{-2/3}\left(\frac{4n}{3}\right)^{-1}\right] + O(n^{-5/3})\right\}.$$

By the mean-value theorem again, we obtain

$$x_{n,k} = \left(\frac{4n}{3}\right)^{1/4} \Omega\left\{\tilde{a}_k \left(\frac{4n}{3}\right)^{-2/3} + 3^{-1/3}2^{-2/3}\left(\frac{4n}{3}\right)^{-1}\right\} + O(n^{-17/12}). \qquad (4.21)$$

To proceed further, we expand $\Omega(\zeta)$ into the Maclaurin series

$$\Omega(\zeta) = \Omega(0) + \Omega'(0)\zeta + \tfrac{1}{2}\Omega''(0)\,\zeta^2 + \cdots. \qquad (4.22)$$

Since $\zeta(1)=0$, we have $\Omega(0)=1$. Differentiating with respect to ζ on both sides of the first equation in (2.6) yields

$$\frac{dw}{d\zeta}=\bar{q}_0^{-1/2}(w)\left\{\frac{3}{2}\int_1^w \bar{q}_0^{1/2}(t)\,dt\right\}^{1/3}=\frac{\zeta^{1/2}}{\bar{q}_0^{1/2}(w)}.$$

Using l'Hôspital's rule, it can be shown that

$$\Omega'(0)=\left.\frac{dw}{d\zeta}\right|_{\zeta=0}=[\bar{q}_0'(1)]^{-1/3}.$$

In a similar manner, it can be verified that

$$\Omega''(0)=\left.\frac{d^2w}{d\zeta^2}\right|_{\zeta=0}=-\frac{1}{5}\frac{\bar{q}_0''(1)}{[\bar{q}_0'(1)]^{5/3}}.$$

Simple computation gives $\bar{q}_0'(1)=18$ and $\bar{q}_0''(1)=114$. Hence

$$\Omega'(0)=\frac{1}{18^{1/3}}\qquad\text{and}\qquad\Omega''(0)=-\frac{1}{5}\frac{114}{18^{5/3}}.$$

Applying (4.22) to (4.21) leads to

$$x_{n,k}=\left(\frac{4n}{3}\right)^{1/4}+\frac{1}{18^{1/3}}\left[\tilde{a}_k\left(\frac{4n}{3}\right)^{-2/3}+3^{-1/3}2^{-2/3}\left(\frac{4n}{3}\right)^{-1}\right]\left(\frac{4n}{3}\right)^{1/4}$$
$$-\frac{1}{10}\frac{114}{18^{5/3}}\left[\tilde{a}_k\left(\frac{4n}{3}\right)^{-2/3}+3^{-1/3}2^{-2/3}\left(\frac{4n}{3}\right)^{-1}\right]^2\left(\frac{4n}{3}\right)^{1/4}+O(n^{-17/12}).$$

To summarize, we have the following result.

THEOREM 3. *Let the zeros of the polynomial $p_n(x)$ in (1.1) be enumerated in decreasing order: $-\infty<x_{n,n}<\cdots<x_{n,2}<x_{n,1}<\infty$. For each positive zero $x_{n,k}$, we have*

$$x_{n,k}=\left(\frac{4n}{3}\right)^{1/4}+\frac{\tilde{a}_k}{18^{1/3}}\left(\frac{4n}{3}\right)^{-5/12}+\frac{1}{6}\left(\frac{4n}{3}\right)^{-9/12}$$
$$-\frac{19\tilde{a}_k^2}{90\cdot 2^{2/3}\cdot 3^{1/3}}\left(\frac{4n}{3}\right)^{-13/12}+O(n^{-17/12}),$$

where $\tilde{a}_k$ is the kth negative zero of the Airy function $Ai(x)$ and the O-symbol depends on k.

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